(Algebraic) Models for (Rational) Homotopy Theory

John Morgan

Higher School for Economics

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References for this lecture

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Plan of Lecture

Today's talk will be discussion of Differential Graded Algebras (DGAs) as an algebraic model for rational homotopy theory.

- Basic definitions and results from homotopy theory including Postnikov towers.
- Rational homotopy theory.
- Homotopy theory for DGAs.
- Comparison of rational homotopy theory and the homotopy theory of DGAs over $\mathbb{Q}$.
- The fundamental group

PART I: HOMOTOPY THEORY

The Homotopy Category has objects as simplicial complexes and morphisms homotopy classes of continuous maps between these.

Remark. One can use CW complexes, or spaces homotpy equivalent with them. In a more algebraic vein one can use simplicial sets as objects with homotopy classes of simplicial maps.

Theorem

Let X and Y be objects of the homotopy category and $f : X \to Y$ a morphsim.
1. If X and Y are connected, f is an isomorphism if and only if f induces an isomorphism on the homotopy groups. (Whitenead's Theorem)

2. If X and Y are simply connected, f is an isomorphism if and only if it induces an isomorphism on homology groups with integer coefficients. (Hurewicz theorem).

Eilenberg-MacLane Spaces

For π an abelian group and $n \geq 1$ an integer, we let $K(\pi, n)$ denote the space, unique up to unique isomorphism, with

$$\pi_i(K(\pi, n)) = \begin{cases} 0 & \text{for} \quad i \neq n \\ \pi & \text{for} \quad i = n \end{cases}.$$

Then there is an element $\iota_\pi \in H^n(K(\pi, n); \pi)$ such that the natural transformation

$$Maps(X, K(\pi, n)) \to H^n(X; \pi)$$

defined by $f \mapsto f^*(\iota_\pi)$ is a bijection.

Remark. The addition on π makes $K(\pi, n)$ an H-space and hence morphisms into $K(\pi, n)$ form a group. The above natural transformation is an isomorphism of groups.

Principal Fibrations with Fiber $K(\pi, n)$

Let $\mathcal{P}$ be the path space of $K(\pi, n+1)$. Then the natural projection $\mathcal{P} \to K(\pi, n+1)$ is a fibration and the fiber is $K(\pi, n)$.

Definition. A fibration $Y \to X$ with fiber $K(\pi, n)$ is said to be *a principal fibration* with fiber $K(\pi, n)$ if it is induced from $\mathcal{P} \to K(\pi, n+1)$ by a map $X \to K(\pi, n+1)$.

Notice that such fibrations are classified by $\mathrm{Mor}(X, K(\pi, n+1)) = H^{n+1}(X; \pi)$. The corresponding cohomology class is called the *k-invariant* of the fibration.

Postnikov Towers

For any simply connected object X there is a tower in the homotopy category, called the *Postnikov Tower* of X:

$$\{\cdots \xrightarrow{p_{n+1}} X_n \xrightarrow{p_n} X_{n-1} \xrightarrow{p_{n-1}} \cdots \xrightarrow{p_3} X_2 \xrightarrow{p_2} X_1 = pt\}$$

and maps $f_n \colon X \to X_n$ with $p_n \circ f_n = f_{n-1}$ for all $n \geq 2$. Furthermore, for every n, the projection $X_n \xrightarrow{p_n} X_{n-1}$ is a principal $K(\pi_n(X), n)$-fibration and $f_n \colon X \to X_n$ induces an isomorphism on all π_i for $i \leq n$.

Remarks. Notice that $\pi_i(X_n) = 0$ for $i > n$. Also, the $\{f_n\}$ determine an isomorphism from X to $\lim\{X_n, p_n\}$. Lastly, the tower is determined by the homotopy groups $\pi_n(X)$ and the k-invariants $k^{n+1} \in H^{n+1}(X_{n-1}; \pi_n(X))$.

PART II: RATIONAL HOMOTOPY THEORY

Why Pass to Rational Homotopy Theory?

Algebraic models for homotopy theory have to be complicated. For example, the homotopy groups of spheres are not known and have no known simple algebraic structure. This is *not* true of the rational homotopy groups of spheres:

$$\pi_i(S^n) \otimes \mathbb{Q} = \begin{cases} 0 & \text{for} \quad i \neq n, \quad \text{for} \quad n \equiv 1 \pmod 2 \\ 0 & \text{for} \quad i \neq n, 2n-1 \text{ for} \quad n \equiv 0 \pmod 2 \\ \mathbb{Q} & \text{in all other cases.} \end{cases}$$

[Recall the Hopf map $S^3 \to S^2$ is homotopically non-trivial and generates $\pi_3(S^2) \cong \mathbb{Z}$.]
Thus, one can hope for simpler algebraic models for rational homotopy theory.

What is a Rational Homology Equivalence?

We restrict to simply connected spaces for this discussion.

First Definition. We say that a map $f : X \to Y$ is a rational equivalence if it induces an isomorphism on the rational homotopy groups.

Second Definition. Replace "rational homotopy" with "reduced rational homology."

Theorem

The first and second definition are equivalent.

The fiber of a rational equivalence (first definition) has torsion homotopy groups. A Serre spectral sequence argument for its Postnikov tower shows that it has trivial reduced rational homology. Thus, the Serre spectral sequence with rational coefficients for f collapses and consequently f induces an isomorphism on rational homology. The converse is proved similarly using the Hurewicz isomorphism.

Rational Homotopy Theory

To define rational homology theory we localize the homotopy category by inverting maps that are rational homotopy equivalence. So an equivalence is a string of morphisms $f_1, f_2, \cdots f_k$ alternating between ordinary morphisms in the forward direction and rational equivalences in the reverse direction. In fact we can do better than this: A morphism $X \to Y$ can always be written as a composition $X \xrightarrow{f} X' \xleftarrow{g} Y$ where g is a rational equivalence.

The basic idea is that every space has a rational equivalence to a *rational* space; i.e., one whose homotopy groups (and reduced homology groups) are rational vector spaces.

Rational Spaces

One way to construct a rational space that is rationally equivalent to X is to replace the Postikov tower for X by its rational version: The $K(\pi_n(X), n)$ get repaced by $K(\pi_n(X) \otimes \mathbb{Q}, n)$ and the k-invariants by their images in rational cohomology.

Another way is to construct a rational n-sphere as the colimit of

$$S^n \xrightarrow{2} S^2 \xrightarrow{3!} S^n \xrightarrow{4!} \cdots .$$

This colimit has reduced homology isomorphic to $\mathbb{Q}$ in degree n and zero in all other dimensions [Recall, that homology commutes with colimits.] The inclusion of S^n into this colimit induces $\mathbb{Z} \to \mathbb{Q}$ on the non-zero homology group and hence is a rational isomorphism. Now given a CW complex one inductively replaces the cells with cones over rational spheres of the same dimension, producing a rational space. There is an inclusion of the original space into this rational space which is a rational equivalence.

Tensoring with $\mathbb{Q}$

Either of the approaches on the previous slide produces the same result which is a functor that tensors a space with $\mathbb{Q}$ and tensors morphisms with $\mathbb{Q}$. Namely, to a simply connected space X we associate $X \otimes \mathbb{Q}$ with a natural map $i_X \colon X \to X \otimes \mathbb{Q}$ which is a rational equivalence. Furthermore, given $f \colon X \to Y$ there is a unique map $f \otimes \mathbb{Q} \colon X \otimes \mathbb{Q} \to Y \otimes \mathbb{Q}$ such that

$$(f \otimes \mathbb{Q}) \circ i_X = i_Y \circ f .$$

One should view this as an analogy with tensoring a nilpotent (or more precisely a pro-nilpotent) group with $\mathbb{Q}$, the so-called Mal'cev completion.
Notice that if f is a rational equivalence, then $f \otimes \mathbb{Q}$ is a homotopy equivalence, and in particular can be inverted up to homotopy.

PART III: HOMOTOPY THEORY OF DIFFERENTIAL GRADED ALGEBRAS

Algebraic Models for Rational Homotopy

Rational models for rational homotopy theory of simply connected spaces go back to Quillen. He introduced several models and showed their equivalence. One of these was differential graded Lie algebras over $\mathbb{Q}$.

It turns out that the models most useful in geometric problems, and most closely connected to the Postnikov tower approach to homotopy theory, are the duals to differential graded Lie algebras, namely DGAs. The theory of these and their direct connection to the rational homotopy category were given by Sullivan.

Basic Definitions for DGAs

Definition

Let K be a field of characteristic zero, always either $\mathbb{Q}, \mathbb{R}$, or $\mathbb{C}$. A differential graded algebra (DGA) over K *is a graded commutative K-algebra A^* with $A^n = 0$ for all $n < 0$. The differential is required to satisfy the Leibniz rule: for homogeneous elements x, y we have $d(xy) = (dx)y + (-1)^{|x|}x(dy)$, where $|x|$ is the degree of x. A morphism of K-DGAs $F \colon A^* \to B^*$ is a degree zero K-linear map preserving the multiplications and d.*

A K-DGA has cohomology which is a graded K-vector space concentrated in degrees ≥ 0. A morphism between DGAs is said to be a *homotopy equivalence* if it induces an isomorphism on cohomology. The homotopy category of DGAs over K is the category obtained by formally inverting homotopy equivalences.

The minimal model of a DGA is its version of a Postnikov tower.

Definition

A DGA $\mathcal{M}$ is minimal *if it is free as a graded commutative algebra on generators of positive degree and if for any $m \in \mathcal{M}$ the element dm is decomposable. A minimal DGA is* simply connected *if it has no generators of degree* 1*.*

For $\mathcal{M}$ a minimal DGA we set $I^*(\mathcal{M})$ equal to the indecomposables $\mathcal{M}^+/\mathcal{M}^+ \wedge \mathcal{M}^+$, and we define $\pi_n(\mathcal{M})$ to be the dual vector space to $I^n(\mathcal{M})$, and we define $\mathcal{M}_n$ to be the subalgebra generated by elements of degrees $\leq n$. The decomposability assumption means that each $\mathcal{M}_n$ is a sub-DGA.

Dual of the Postnikov tower

We consider a simply connected minimal DGA $\mathcal{M}$ with a finite dimensional space of indecomposables in each degree.. Since there are no generators of degree 1 it follows that if x is an element of degree n, then $dx \in \mathcal{M}_{n-1}$. Let V_n be a subspace of elements of degree n that projects isomorphically onto the indecomposables in degree n. Then as an algebra $\mathcal{M}_n = \mathcal{M}_{n-1} \otimes \Lambda^*(V_n)$ and the differential on $\mathcal{M}_n$ is determined by the fact that it extends the differential on $\mathcal{M}_{n-1}$ and the homomorphism $d \colon V_n \to \mathcal{M}_{n-1}$ whose image is contained in the closed elements of degree $n+1$. Thus, it induces a map $V_n \to H^{n+1}(\mathcal{M}_{n-1})$ or equivalently an element in $H^{n+1}(\mathcal{M}_{n-1}; V_n^*) = H^{n+1}(\mathcal{M}_{n-1}; \pi_n(\mathcal{M}))$.

Dual of the Postnikov tower

The isomorphism class of $\mathcal{M}_n$ is determined by $\mathcal{M}_{n-1}$ and the resulting element in $H^{n+1}(\mathcal{M}_{n-1}; \pi_n(\mathcal{M}_{n-1}))$. This is the k-invariant of the extension.

- Show that a map between simply connected minimal DGAs is an isomorphism if and only if it induces an isomorphism on the homotopy groups.
- Let $\mathcal{M}_{n-1}$ be a minimal DGA generated in degrees $\leq n-1$ and let $\mathcal{M}'$ and $\mathcal{M}''$ be minimal algebras generated in degrees $\leq n$, each of which has $\mathcal{M}_{n-1}$ as the subalgebra generated in degrees $\leq n-1$. Show that the identify map of these subalgebras extends to an isomorphism $\mathcal{M}' \to \mathcal{M}''$ if and only if there is an isomoprhism $I^n(\mathcal{M}') \to I^n(\mathcal{M}'')$ transforming the n^{th} k-invariant of $\mathcal{M}'$ to that of $\mathcal{M}''$.

Minimal Models for DGAs

Definition

A minimal model *for a DGA A^* is a minimal DGA $\mathcal{M}$ together with a homotopy equivalence $\rho\colon \mathcal{M} \to A^*$*

Theorem

Let A^* be a simply connected DGA. Then there is a minimal model for A^*, $\rho\colon \mathcal{M} \to A^*$. Given two minimal models $\rho\colon \mathcal{M} \to A^*$ and $\rho'\colon \mathcal{M}' \to A^*$ there is an isomorphism $\psi\colon \mathcal{M} \to \mathcal{M}'$ and a homotopy from ρ to $\rho' \circ \psi$, where by a homotopy we mean a morphism $\mathcal{M} \to A^* \otimes \Lambda(t, dt)$ whose restriction to $t = 0$ is ρ and whose restriction to $t = 1$ is $\rho' \circ \psi$.

Inductive Construction of the Minimal model

Let A^* be a simply connected DGA. Set $\mathcal{M}_2 = \Lambda^*(H^2(A))$ with $d = 0$, and define $\rho_2 \colon \mathcal{M}_2 \to A^*$ by choosing a linear splitting of $H^2(A)$ back to the closed elements of A^2 and extending multiplicatively. This map induces an isomorphism on H^2 and an injection on H^3. Now let $V_3 = H^4(\mathcal{M}_2, A^*)$ and choose a splitting s of V_3 back to the relative cycles (x, a) of degree 4 (meaning that $x \in (\mathcal{M}_2)^4$, $a \in A^3$ and $da = \rho_2(x)$). We form $\mathcal{M}_3 = \mathcal{M}_2 \otimes \Lambda^*(V_3)$ and extend d by requiring $d(v) = p_1(s(v))$ where p_1 is projection onto the first factor. We define $\rho_3 \colon \mathcal{M}_3 \to A^*$ to extend ρ_2 and map v to $p_2(s(v))$. One checks easily that this map of DGAs induces an isomorphism on H^* for $* \leq 3$ and an injection on H^4.

The analogous argument proceeds inductively over increasing degrees. The minimal model for A^* consists of the union of the $\mathcal{M}_n$ and the union of the maps ρ_n.

Functorality of the Minimal Model up to Homotopy

Theorem

Suppose that $F\colon A^* \to B^*$ is a morphism of simply connected DGAs. Let $\rho_A\colon \mathcal{M}_A \to A^*$ and $\rho_B\colon \mathcal{M}_B \to B^*$ be minimal models. Then there is a diagram commutative up to homotopy

$$\begin{tikzcd} A^* \arrow[r, "F"] & B^* \\ \mathcal{M}_A \arrow[u, "\rho_A"'] \arrow[r, "F'"] & \mathcal{M}_B \arrow[u, "\rho_B"'] \end{tikzcd}$$

The map F' is well defined up to homotopy.

A map between minimal DGAs induces a map on the indecomposables and homotopic maps between minimal DGAs induce the same map of indecomposables. It follows that the homotopy groups of a DGA are well defined up to unique isomorphism and are (contravariantly) functorial.

PART IV: COMPARISON OF HOMOTOPY THEORY OF SPACES AND DGAs

The $\mathbb{Q}$-DGA associated to a simplex

Let Δ^n be an n-simplex embedded in $\mathbb{R}^{n+1}$ with coordinates $(t_0, \ldots, t_n)$ as

$$\{(t_0, \ldots, t_n) \big| \sum t_i = 1 \quad \text{and} \quad t_i \geq 0 \quad \text{for all} \quad i\}.$$

We define a $\mathbb{Q}$-DGA associated to Δ^n by:

$$\Lambda^*(\Delta^n) = \Lambda^*(t_0, \ldots, t_n, dt_0, \ldots, dt_n)/(\sum_i t_i = 1; \sum_i dt_i = 0).$$

By this we mean the tensor product of the polynomial algebra on the t_i in degree 0 with the exterior algebra on the dt_i in degree 1 modulo the given relations. An elementary exercise shows

Theorem

$H^*(\Lambda^*(\Delta^n))$ is zero in all degrees except degree 0 where it is isomorphic to $\mathbb{Q}$.

Notice that if τ is a face of σ then there is a natural restriction map of DGAs

$$\Lambda^*(\sigma) \to \Lambda^*(\tau).$$

Definition

For a simplicial complex X we define the DGA of piecewise lilnear (pl) differential forms of X, *denoted $\Lambda^*(X)$, to be*

$$\Lambda^*(X) = \Big\{ \{\omega_\sigma\} \Big| \text{ for any } \tau < \sigma \text{ we have } \omega_\sigma|_\tau = \omega_\tau \Big\}.$$

Here σ runs over the simplices of X.

It is easy to see that the DGA sturctures on the forms on each simplex determine the DGA structure on the pl differential forms on X.

The $\mathbb{Q}$-DGA associated to a simplicial complex

Associating to a simplicial complex X the DGA $\Lambda^*(X)$ determines a functor from the category of simplicial complexes and simplicial maps to the category of DGAs over $\mathbb{Q}$.

Theorem

Integration of pl differential forms over simiplicial cycles determines an isomorphism natural under simplicial maps

$$H^*(\Lambda^*(X)) \to \mathrm{Hom}(H_*(X;\mathbb{Z}), \mathbb{Q}) = H^*(X;\mathbb{Q}).$$

The basic points are that the cohomology of the DGA of a simplex is trivial and the restriction maps from forms on a complex to forms on a subcomplex are surjective. With these, the result is easily proved by induction on the number of simplices for finite complexes and then by limiting to all complexes.

Theorem

Let X be a simply connected space. Then the homotopy groups of $\Lambda^*(X)$ are identified with $\pi_*(X) \otimes \mathbb{Q}$ and the cohomology of $\Lambda^*(X)$ is identified with $H^*(X;\mathbb{Q})$. Let $\mathcal{M}$ be a minimal model for $\Lambda^*(X)$. For each n, the sub-DGA $\mathcal{M}_n$ is a minimal model for $\Lambda^*(X_n)$ where X_n is the n^{th}-stage of the Postnikov tower for X and the k-invariant describing the extenstion $\mathcal{M}_{n-1} \subset \mathcal{M}_n$, which is an element of $H^{n+1}(\mathcal{M}_{n-1};\pi_n(\mathcal{M}))$ is identified with the image after tensoring with $\mathbb{Q}$ of the k-invariant of the prinicpal fibration $X_n \to X_{n-1}$ which lies in $H^{n+1}(X_{n-1};\pi_n(X))$.

In particular, the minimal model for $\Lambda^*(X)$ is equivalent to the Postnikov tower of $X \otimes \mathbb{Q}$.

We can define the *real homotopy type* of a simply connected simplicial X to be the homotopy type of $\Lambda^*(X) \otimes \mathbb{R}$. If $\mathcal{M}_X$ is a minimal model for $\Lambda^*(X)$ then the real homology theory is given by $\mathcal{M}_X \otimes \mathbb{R}$. In particular, the homotopy groups are $\pi_*(X) \otimes \mathbb{R}$.
Warning. In general one cannot reconstruct the rational homotopy type from the real homotopy type.
As an example, consider CW complexes which are a union of two 2-cells and one 4. The attaching map $S^3 \to S^2 \vee S^2$ is determined by an integer symmetric 2×2 matrix. This matrix, up to rational equivalence and rational scale, is a complete invariant of the homotopy type. The image of the determinant of the matrix in $\mathbb{Q}^*/(\mathbb{Q}^*)^2$ is an invariant of the rational homotopy type and takes all values in this infinite group of exponent 2. The only invariant of the real homotopy type is the absolute value of the signature.

Theorem

Let M be a smooth manifold. Then the real homotopy type of M is homotopy type of the deRham complex of M.

Fix a smooth triangulation of M. Let X be the underlying simplicial complex of this triangulartion. Let $\Omega^*(M)$ be the deRham complex of M, and let $\Omega^*_{\text{piecewise } C^\infty}(M)$ be the piecewise C^∞-forms on this smooth triangulation. We have the natural inclusion of $\Omega^*(M) \to \Omega^*_{\text{piecewise } C^\infty}(M)$. By arguments similar to those in the case of pl forms, one shows that this map induces an isomorphism on cohomology as does the inclusion $\Lambda^*(X) \otimes \mathbb{R}$.

Remark. Similarly, there is the complex homotopy type and an analogous result for complex differential forms on a smooth manifold.

PART V: THE CASE OF THE FUNDAMENTAL GROUP

The nilpotent completion of a group

Associated to a group G is the lower central seies:

$$G = G_0 \supset G_1 \supset G_2 \supset \cdots$$

where $G_1 = [G, G]$ is the commutator subgroup and $G_n = [G, G_{n-1}]$ for all $n \geq 2$. For each $n \geq 1$ we set $N_n = G/G_n$. The N_n are nilpotent groups, with $N_1 = A$ being the abelianization of G. This tower of nilpotent groups associated to G

$$A \longleftarrow N_2 \longleftarrow N_3 \longleftarrow \cdots$$

whose limit is the *nilpotent completion* of G. The quotient map $G \to N_n$ is universal for maps from G to nilpotent groups of order of nilpotency n.

The Mal'cev completion of a niltpotent group

A nilpotent group N of order of nilpotency k is the fundamental group of a Postnoikov tower

$$X_k \to X_{k-2} \to \cdots \to X_2 \to X_1 \to \{pt\}$$

where each $X_r \to X_{r-1}$ is a principal fibration with fiber an Eilenberg-MacLane space $K(A_r, 1)$ for some abelian group A_r. Thus, as before we can tensor this tower with $\mathbb{Q}$ producing a space whose fundamental group is a torsion-free, divisible nilpotent group, one whose successive quotients are rational vector spaces. This fundamental group is the Mal'cev completion of N and it is fair to denote it $N \otimes \mathbb{Q}$. Let $\mathcal{N}$ be the tower of nilpotent rational Lie algebras determined by the tower of nilpotent groups. (These are equivalent data by the Baker-Campbell Hausdorff formula.)

The fundamental group of a connected DGA

Let A^* be a DGA with $H^0(A)$ equal to the ground field K. We construct a 1-minimal model for A^* as follows. Begin with $\mathcal{M}_1 = \Lambda^*(H^1(A))$ $(d = 0)$ and a map $\rho_1 \colon \mathcal{M}_1 \to A^*$ inducing an isomorphism on H^1. Let $V_2 \subset H^2(\mathcal{M}_1) = (\mathcal{M}_1)^2$ be the kernel of the map $\rho_1^* \colon H^2(\mathcal{M}_1) \to H^2(A^*)$. Define $\mathcal{M}_2 = \mathcal{M}_1 \otimes \Lambda^*(V_2)$ where $d \colon V_2 \to (\mathcal{M}_1)^2$ is the inclusion. We extend the map $\mathcal{M}_1 \to A^*$ to a map $\mathcal{M}_2 \to A^*$ by defining $\rho_2 \colon V_2 \to A^1$ solving the equation $d_A \rho_2 = \rho_1 d_{\mathcal{M}_2}$.

The fundamental group of a connected DGA

We repeat this construction *ad infinitum* producing an increasing sequence of DGAs generated in degree one:

$$\mathcal{M}_1 \subset \mathcal{M}_2 \subset \mathcal{M}_3 \subset \cdots$$

all mapping compatibly to A^* such that the induced map on the union $\mathcal{M}$, a DGA generated in degree one that maps to A^* inducing an isomorphism on H^1 and an injection on H^2.
Let $\mathcal{L} = \mathcal{L}_1 \longleftarrow \mathcal{L}_2 \longleftarrow \cdots$ be the projective system of vector spaces dual to increasing union of spaces of generators for the $\mathcal{M}_n$.
Then the differential on the $\mathcal{M}_n$ is dual to a skew-symmetric bracket for $\mathcal{L}_n$ and the Bianchi identity for this bracket is dual to $d^2 = 0$. In this way we dualize the increasing union of minimal rational DGAs generated in degree 1 to a tower of rational nilpotent Lie algebras. These can be integrated to a tower of rational nilpotent groups, and the limit of these groups is defined to be the fundamental group of A^*.

Forms and the nilpotent completion of the fundamental group

Analogous to the results for simply connected spaces we have:

Theorem

Let X be a simplicial complex and then the fundamental group of $\Lambda^*(X)$ is naturally identified with the Mal'cev completion of $\pi_1(X)$.

Let X be a smooth triangualtion of a smooth manifold M. The homotopy equivalence of $\Lambda^*(X) \otimes \mathbb{R}$ with the deRham complex of M implies:

Corollary

Let M be a smooth manifold Then the fundamental group of the deRham complex $\Omega^*(M)$ is naturally identified with the tensor product of the Mal'cev completion of the fundamental group of M with $\mathbb{R}$.