Mirror symmetry for Higgs bundles I

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NONABELIAN HODGE THEORY

- Σ compact Riemann surface $g > 1$

- G Lie group

- $H^1(\Sigma, G)$ = equivalence classes of flat G-bundles

- ... apply Hodge theory

THE ABELIAN CASE

- $H^1(\Sigma, \mathbf{C}^*) = H^1(\Sigma, \mathbf{C})/H^1(\Sigma, \mathbf{Z}) \cong (\mathbf{C}^*)^{2g}$ (holonomy)

- Hodge theory: $H^1(\Sigma, \mathbf{C}) = H^{10} \oplus H^{01}$

- flat connection $d + \alpha^{10} + \alpha^{01}$

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- flat connection $d + \alpha^{10} + \alpha^{01}$

- $\beta, \gamma \in H^{10}$, flat $U(1)$-connection $\nabla_A = d + \beta - \bar{\beta}$

- $\nabla_A + \gamma + \bar{\gamma} = d + \alpha^{10} + \alpha^{01}$

  if $\beta + \gamma = \alpha^{10}, -\beta + \gamma = \overline{\alpha^{01}}$

- $d + \beta - \bar{\beta} \sim \bar{\partial} - \bar{\beta}$ holomorphic structure

- flat $\mathbf{C}^*$-connection $\cong$

  holomorphic line bundle + holomorphic 1-form

- $H^1(\Sigma, \mathbf{C}^*) \cong \mathrm{Jac}(\Sigma) \times H^0(\Sigma, K)$

- $(\gamma, \nabla_A) \in H^0(\boldsymbol{\Sigma}, K) \times \mathrm{Jac}(\boldsymbol{\Sigma}) \cong H^1(\boldsymbol{\Sigma}, \mathbf{C}^*)$ *not holomorphic*
- $\int_{\Sigma} \beta \wedge \bar{\beta} + \gamma \wedge \bar{\gamma}$ flat metric
- $\int_{\Sigma} \bar{\beta} \wedge \gamma$ holomorphic symplectic structure

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- $\displaystyle\int_{\Sigma} \bar{\beta} \wedge \gamma$ holomorphic symplectic structure

- hyperkähler metric, complex structures I, J, K

  $I^2 = J^2 = K^2 = IJK = -1$

- I: $H^0(\boldsymbol{\Sigma}, K) \times \mathrm{Jac}(\boldsymbol{\Sigma}) \cong T^*\mathrm{Jac}(\boldsymbol{\Sigma})$

  J: $H^1(\boldsymbol{\Sigma}, \mathbf{C}^*) \cong (\mathbf{C}^*)^{2g}$

- $0 \to H^1(\mathbf{\Sigma}, \mathbf{R}) \to H^1(\mathbf{\Sigma}, \mathbf{R}^*) \to \mathbf{Z}_2^{2g} \to 0$

- $\alpha^{10} + \overline{\alpha^{10}} \in H^1(\mathbf{\Sigma}, \mathbf{R})$

- $p : H^0(\mathbf{\Sigma}, K) \times \mathrm{Jac}(\mathbf{\Sigma}) \to H^0(\mathbf{\Sigma}, K)$

- $H^1(\mathbf{\Sigma}, \mathbf{R}^*) = 2^{2g}$ *holomorphic* sections of p

  $H^1(\mathbf{\Sigma}, \mathbf{R}^*) =$ *real* points of $(\mathbf{C}^*)^{2g}$

HIGGS BUNDLES

- $H^1(\Sigma, GL(n, \mathbf{C})) \sim \mathrm{Hom}(\pi_1(\Sigma), GL(n, \mathbf{C}))/GL(n, \mathbf{C})$

- flat rank n vector bundle V

- irreducible reps $\Rightarrow$ smooth complex manifold

- $H^1(\Sigma, GL(n,\mathbf{C})) \sim \mathrm{Hom}(\pi_1(\Sigma), GL(n,\mathbf{C}))/GL(n,\mathbf{C})$

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- Hodge theory $\Rightarrow$ represent as $\nabla_A + \Phi + \Phi^*$

  ∇_A unitary connection, $\Phi \in H^0(\Sigma, \mathrm{End}\, V \otimes K)$

  (Corlette, Donaldson, Simpson, NJH,...)

- Higgs field: $\Phi \in H^0(\Sigma, \mathrm{End}\, V \otimes K)$

- $\nabla_A + \Phi + \Phi^*$ flat $\Rightarrow F_A + [\Phi, \Phi^*] = 0$

  Higgs bundle equations

- stability $\Rightarrow$ existence

- moduli space $\mathcal{M}$ hyperkähler

- Kähler forms $\omega_1, \omega_2, \omega_3$ for complex structures I, J, K

- holomorphic symplectic form $\omega^c = \omega_2 + i\omega_3$ relative to I etc.

- $\Rightarrow$ covariant constant holomorphic volume form

Calabi-Yau

MIRROR SYMMETRY

- symplectic geometry: A-model
- complex geometry: B-model
- $+$ Calabi-Yau

ELSEVIER

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NUCLEAR PHYSICS B

Mirror symmetry is T-duality

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SYZ MIRROR SYMMETRY

- Calabi-Yau manifold M^n: ω symplectic form,

  Ω holomorphic n-form

- special Lagrangian fibration: $p : M \to B$

  ($\omega, \mathrm{Re}\,\Omega$ vanish on fibres)

- fibres are tori T_b

- mirror = dual fibration, fibre over b = moduli space of flat $U(1)$-bundles over T_b

- $X \in T_b B \sim$ section of normal bundle of T_b
- $i_X \omega$ well-defined and closed on T_b since fibre Lagrangian
- $[i_X \omega] \in H^1(T_b, \mathbf{R})$
- $p^* TB \cong T_F^*$

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- $[i_X \omega] \in H^1(T_b, \mathbf{R})$

- $p^* TB \cong T_F^*$

- homology class $A \in H_1(T_b, \mathbf{R}) \Rightarrow$ closed 1-form a on B

  .... local flat coordinates $a_i = dx_i$

- $i_X\Omega$ well-defined and closed since fibre special Lagrangian

- $[i_X\Omega] \in H^{n-1}(T_b, \mathbf{R}) \cong (H^1(T_b, \mathbf{R}))^*$

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- $[i_X\Omega] \in H^{n-1}(T_b, \mathbf{R}) \cong (H^1(T_b, \mathbf{R}))^*$

- $p^*TB \cong T_F$ and $p^*TB \cong T_F^* \Rightarrow$ metric on B

- Hessian form $\quad g_{ij} = \dfrac{\partial^2 \phi}{\partial x_i \partial x_j}$

flat coordinates

- $\widehat{M}$ dual fibration

- Gauss-Manin connection $T\widehat{M} = T_F \oplus p^* TB = H^1(\widehat{T}_b) \oplus H^1(\widehat{T}_b)$

- $T\widehat{M} = H^1(\widehat{T}_b) \oplus iH^1(\widehat{T}_b)$ complex

- metric $\Rightarrow$ Kähler metric on $\widehat{M}$

- semi-flat metric –

  Lagrangian fibres are orbits of Killing vector fields

ISSUES

- Compact Calabi-Yau threefold with a SL fibration?

- (dimension 2 – elliptic K3 – hyperkähler)

- semi-flat metric $\sim$ large complex structure limit?

THE INTEGRABLE SYSTEM

- algebraic curve Σ

- holomorphic G^c-principal bundle P

- section $\Phi \in H^0(\Sigma, \mathfrak{g} \otimes K)$ = Higgs field

- $+$ stability condition

- $\Rightarrow$ reduction to maximal compact G
- $\Rightarrow$ G-connection A
- $F_A + [\Phi, \Phi^*] = 0$
- moduli space hyperkähler

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- holomorphic symplectic manifold

THE FIBRATION

- hyperkähler moduli space $\mathcal{M}(G^c)$

  $\dim_{\mathbf{C}} = 2(g-1)\dim G^c$

- principal G^c-bundle, $\Phi \in H^0(\Sigma, \mathfrak{g} \otimes K)$

- invariant polynomials $p_1, \ldots, p_\ell$ on $\mathfrak{g}$

  $p_m(\Phi) \in H^0(\Sigma, K^{d_m})$

- fibration $\mathcal{M}^{2k} \to \mathbf{C}^k$

- these k functions Poisson-commute

- integrable system

- generic fibre abelian variety A

- $G^c = GL(n, \mathbf{C})$ $\det(x - \Phi) = 0$ spectral curve S

- fibre = $\mathrm{Jac}(S)$

- $\pi : S \to \Sigma$
- line bundle L on S, $V = \pi_* L$ direct image
- x section of $\pi^* K$, $x : L \to L\pi^* K$
- $\Phi = \pi_* x$

- Hyperkähler manifold + holomorphic Lagrangian fibration wrt I

- $\Rightarrow (\omega_2 + i\omega_3)$ vanishes on fibres

- ω_2 vanishes and real or imaginary part of

  $(\omega_3 + i\omega_1)^k$ vanishes

- $\Rightarrow$ special Lagrangian fibration for complex structure J

$$\omega = \omega_2, \quad \Omega = (\omega_3 + i\omega_1)^k$$

- $\mathcal{M}$ non-compact

- instead of large complex structure limit

- ... consider $\|\Phi\|^2 \to \infty$

- hyperkähler metric is approximated by the semi-flat metric

(R.Mazzeo, J.Swoboda, H.Weiss, F.Witt, *Asymptotic geometry of the Hitchin metric*, arXiv:1709.03433)

- Higgs bundle moduli space $p : \mathcal{M} \to B$

- *generic* fibre smooth abelian variety

- $B^{\mathsf{reg}} \sim$ smooth spectral curve

- approximation over B^{reg} as $\|\Phi\| \to \infty$

THE SEMI-FLAT METRIC

- M complex symplectic manifold

- $p : M \to B$ holomorphic Lagrangian fibration

- ... base B has a "special Kähler structure"

- M complex symplectic manifold
- $p : M \to B$ holomorphic Lagrangian fibration
- ... base B has a "special Kähler structure"
- flat affine connection ∇ such that ...

  (i) Kähler form ω is covariant constant

  (ii) complex structure $I \in T \otimes T^*$ is locally ∇X

- special Kähler structure defines a hyperkähler metric on M

- $$\begin{aligned}\omega_1 &= \sum \frac{\partial^2\phi}{\partial x_j \partial x_k} dx_j \wedge dy_k \\ \omega_2 + i\omega_3 &= -\frac{1}{2}\sum d(x_j + iy_j) \wedge d(x_k + iy_k)\end{aligned}$$

$$K = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \qquad J = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$$

- determined by variation of period matrix

- deformations of compact holomorphic Lagrangians

- $\Rightarrow$ special Kähler structure on moduli space

- B^{reg} = space of smooth spectral curves S

  $\det(x-\Phi) = x^n + a_1x^{n-1} + \ldots + a_n = 0$

- $x \in H^0(S, \pi^*K) \Rightarrow$ curve in the total space of K

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- $x \in H^0(S, \pi^* K) \Rightarrow$ curve in the total space of K

- cotangent bundle: curve in a symplectic surface is Lagrangian

- canonical holomorphic 1-form θ on $T^*\Sigma$ ($\theta \sim x$)

- potential for special Kähler metric on space of curves:

$$K = -\frac{i}{4}\int_S \theta \wedge \bar{\theta}$$

- this defines the semi-flat metric on $\mathcal{M}$

(D.Baraglia & Z.Huang, *Special Kähler geometry of the Hitchin system and topological recursion*, arXiv:1707.04975v2)

LANGLANDS DUALITY

Invent. math. 153, 197–229 (2003)
DOI: 10.1007/s00222-003-0286-7

Inventiones mathematicae

Mirror symmetry, Langlands duality, and the Hitchin system

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Abstract. Among the major mathematical approaches to mirror symmetry are those of Batyrev-Borisov and Strominger-Yau-Zaslow (SYZ). The first is explicit and amenable to computation but is not clearly related to the physical motivation; the second is the opposite. Furthermore, it is far from obvious that mirror partners in one sense will also be mirror partners in the other. This paper concerns a class of examples that can be shown to satisfy

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Electric-Magnetic Duality And The Geometric Langlands Program

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- $\mathcal{M}(G)$ Higgs bundle moduli space

- hyperkähler

- holomorphic Lagrangian fibration

- ....its mirror is $\mathcal{M}({}^LG)$ where ${}^LG$ is the Langlands dual group

R.Donagi & T.Pantev, *Langlands duality for Hitchin systems,* Invent. math. **189** (2012), 653–735.

- ${}^{L}GL(n,\mathbf{C}) = GL(n,\mathbf{C})$

- (Jac(S) dual to itself)

- ${}^{L}SL(n,\mathbf{C}) = PGL(n,\mathbf{C})$

- generic fibre for $SL(n,\mathbf{C})$ = Prym variety $P(S,\Sigma)$

- $$0 \to P(S,\Sigma) \to \mathsf{Jac}(S) \stackrel{\mathsf{Nm}}{\to} \mathsf{Jac}(\Sigma) \to 0$$

$$0 \to \mathsf{Jac}(\Sigma) \stackrel{\pi^*}{\to} \mathsf{Jac}(S) \to P(S,\Sigma)^\vee \to 0$$

- $$\begin{aligned} P(S,\Sigma)^\vee &\cong (P(S,\Sigma)/P(S,\Sigma) \cap \mathsf{Jac}(\Sigma)) \\ &= P(S,\Sigma)/\pi^* H^1(\Sigma, \mathbf{Z}_n) \end{aligned}$$

- abelian variety A, $A^\vee \cong A/\Gamma$

$Sp(2m, \mathbf{C})$ AND $SO(2m+1, \mathbf{C})$

$Sp(2m, \mathbf{C})$

- E rank $2m$ symplectic vector bundle

- $\Phi \in H^0(\Sigma, \mathfrak{g} \otimes K) = H^0(\Sigma, S^2 E \otimes K)$

- eigenvalues $\pm\lambda_i$

- spectral curve $S \subset |K|$:

  $0 = \det(x - \Phi) = x^{2m} + a_2 x^{2m-2} + \cdots + a_{2m}$

- involution $\sigma(x) = -x$

- $\pi : S \to \Sigma$

- $E = \pi_* L$

- $x : L \to L \otimes \pi^* K$ and $\Phi = \pi_* x$

- .... where $L = U \pi^* K^{m-1/2}$ and $\sigma^* U \cong U^*$

- abelian variety = Prym variety $P(S, \bar{S})$ $(\bar{S} = S/\sigma)$

$SO(2m+1,\mathbf{C})$

- V rank $(2m+1)$ orthogonal vector bundle

- $\Phi \in H^0(\Sigma, \mathfrak{g} \otimes K) = H^0(\Sigma, \Lambda^2 V \otimes K)$

- eigenvalues $0 \pm \lambda_i$

$SO(2m+1, \mathrm{C})$

- V rank $(2m+1)$ orthogonal vector bundle

- $\Phi \in H^0(\Sigma, \mathfrak{g} \otimes K) = H^0(\Sigma, \Lambda^2 V \otimes K)$

- eigenvalues $0 \pm \lambda_i$

- kernel $\sim \Phi^m \in \Lambda^{2m} V \otimes K^m \cong V \otimes K^m$

- reducible spectral curve

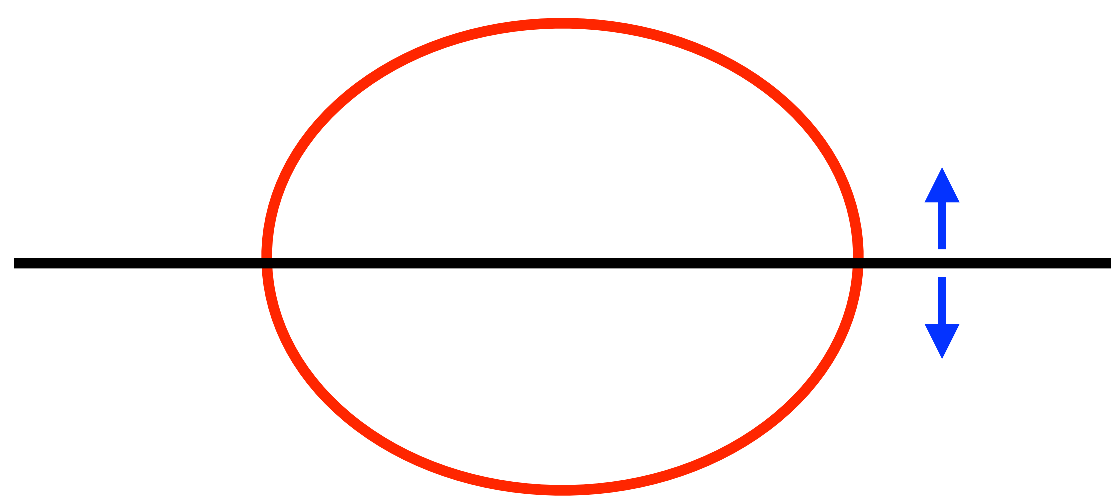
  $$0 = \det(x - \Phi) = x(x^{2m} + a_2 x^{2m-2} \overset{S}{+} \cdots + a_{2m})$$

- $V = \pi_* L$ where ...

- on $x^{2m} + a_2 x^{2m-2} + \cdots + a_{2m} = 0$

  $L = U\pi^* K^m$ and $U \in P(S, \bar{S})$

- on $x = 0 \cong \Sigma$, $L = K^m$



$x = 0$

$x^{2m} + a_2 x^{2m-2} + \cdots + a_{2m} = 0$

- $x = 0$ fixed point set of σ

- $\sigma^* U \cong U^* \Rightarrow$ trivialization of U^2 on $x = 0$

- $K^m \cong UK^m \Rightarrow \pm 1$ choice

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- $x = 0 \Leftrightarrow a_{2m}(z) = 0$: $4m(g-1)$ points

- $2^{4m(g-1)}$ covering of $P(S, \bar{S})$

- overall ± 1

  $+$

  $SO(2m+1)$-bundle spin/non-spin

- $2^{4m(g-1)-2}$ covering of $P(S, \bar{S})$

- $= P(S, \bar{S})/p^* H^1(\bar{S}, \mathbf{Z}_2)$

  $=$ dual of $P(S, \bar{S})$

BRANES

- symplectic geometry:

  A-brane = Lagrangian submanifold + flat connection

- complex geometry:

  B-brane = complex submaniifold + holomorphic bundle

- .... and generalizations, sheaves etc.

- hyperkähler: complex structures I, J, K

- symplectic forms $\omega_1, \omega_2, \omega_3$

- BAA-brane = holomorphic Lagrangian submanifold wrt I + flat connection

- BBB-brane = hyperkähler submanifold + hyperholomorphic bundle

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- BAA-brane = holomorphic Lagrangian submanifold wrt I + flat connection
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mirror symmetry is supposed to interchange these